

A new proof of normalization for **NS4**

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Abstract

In 1965 Dag Prawitz presented an extension of Gentzen-type systems of Natural Deduction to modal concepts, obtaining three new systems of Natural Deduction for S4. Maria da Paz Medeiros showed in 2006 that the proof of normalization for classical S4 does not hold and proposed a new proof of normalization for a logically equivalent system, the NS4. However, two problems in the proof of a lemma used by Medeiros in her proof were presented by Andou Yuuki in 2010. This paper introduces a new proof of normalization for NS4.

1 Introduction

In his doctoral dissertation, Dag Prawitz extended the Gentzen-type systems of Natural Deduction (ND) [2] to modal concepts, obtaining three Gentzen-type systems of ND for S4 based on classical, intuitionistic and minimal predicate logic. For this purpose, Prawitz added a modal operator for necessity (the “N” operator, here represented by \Box) and defined the following rules for introduction and elimination of \Box :

$$\frac{A}{\Box A} \Box\text{-I} \qquad \frac{\Box A}{A} \Box\text{-E} \qquad (1)$$

Prawitz noticed that with these new rules the normalization procedure does not hold. He presented three versions of those modal systems of which only the third one would accept the Normalization Theorem.

About forty years later, Medeiros [1] provided a counterexample showing that the Normalization Theorem does not hold even on Prawitz’s third version of the ND system for classical S4. Medeiros also proposed a correction to Prawitz’s system.

However, recently Yuuki [3] pointed out two problems in the proof of a lemma (the critical lemma) used in the proof of Medeiros' Normalization Theorem.

Our goal on this paper is to present a correction of Medeiros's proof of the aforementioned lemma. In section 2 we outline the original Prawitz's system for classical S4, and the counterexamples by Medeiros are stated at section 3. At section 4 we discuss the two cases in which the system may not produce valid derivations on NS4 due to problems in the proof of the critical lemma. In section 5 we introduce a new proof of the critical lemma.

2 Prawitz's S4 classical modal logic system

The difference between each one of the S4 classical modal logic systems presented by Prawitz is on the restriction of the \Box -I rule. The first version, has the restriction that \Box -I can be applied only when its premisses depend solely on modal formulas. However, the Normalization Theorem does not hold. For instance, Prawitz pointed out that the derivation

$$\begin{array}{c}
 \frac{\frac{\frac{[\Box A]^2}{A} \quad \frac{[\Box B]^1}{B}}{A \wedge B}}{\Box(A \wedge B)}^1 \\
 \frac{\frac{\frac{\Box A \wedge \Box B}{\Box B} \quad \frac{\frac{\Box A \wedge \Box B}{\Box A} \quad \frac{\frac{\Box B \rightarrow \Box(A \wedge B)}{\Box A \rightarrow (\Box B \rightarrow \Box(A \wedge B))}^1}}{\Box B \rightarrow \Box(A \wedge B)}^2}{\Box(A \wedge B)}^2
 \end{array} \quad (2)$$

is a derivation in this system but its reduction

$$\begin{array}{c}
 \frac{\frac{\Box A \wedge \Box B}{\Box A} \quad \frac{[\Box B]^1}{B}}{A \wedge B}^* \\
 \frac{\frac{\Box A \wedge \Box B}{\Box B} \quad \frac{\frac{\Box(A \wedge B)}{\Box B \rightarrow \Box(A \wedge B)}^1}}{\Box(A \wedge B)}^1
 \end{array}$$

is not. Notice that \Box -I cannot be applied in $*$ for it depends on a formula that is not modal ($\Box A \wedge \Box B$).

Hence Prawitz liberalized the restriction on \Box -I, also allowing it to be applied when its premisses depend only on essentially modal formulas. A

formula A is *essentially modal* when “each occurrence of a predicate parameter or predicate constant in A stands within the scope of an occurrence of” \Box .

This notion clears the way to a second version of the system, in which both (2) and its reduction are correct derivations. However, we can still find a derivation whose reduction is not a derivation in the system:

$$\frac{\frac{\Box A \wedge B}{\Box A} \quad \frac{\frac{[\Box A]^1}{\Box \Box A} \quad \Box A \rightarrow \Box \Box A}{\Box \Box A} 1}{\Box \Box A} \triangleright \frac{\frac{\Box A \wedge B}{\Box A} \quad \Box \Box A}{\Box \Box A} *$$

Notice that \Box -I could not be applied in $*$ for $\Box A \wedge B$ is not an essentially modal formula.

Finally, Prawitz’s last restriction states that, if a formula A depends on an assumption B and there exists an essentially modal formula F on the thread of A from B such that A depends on every assumption which F depends on, then \Box -I could be applied at A .

3 A counterexample for Prawitz’s classical S4 system

In what follows, $\frac{\vec{F}}{\Box \vec{B}}$ is a sequence of deductions of the form $\frac{\vec{\Pi}_i}{\Box \vec{B}_i}$, where no two $\Box B_i$ ’s are equal. The reason of this restriction is explained in item 4 of the proof of the critical lemma and it does not affect the completeness of the system. We also write $[\Box \vec{B}]^k$ to indicate that each $\Box B_i$ is discharged at k . The labels may be dropped.

Medeiros [1] showed that Prawitz’s last version of **S4** would not avoid maximal formulas¹ by presenting the following derivation

$$\frac{\frac{[\neg \Box A]^1 \quad \neg \Box A \rightarrow B}{B} \quad \neg B}{\frac{\frac{\perp}{\Box A} 1}{A} \quad \frac{C \rightarrow A}{\Box(C \rightarrow A)}}$$

which is valid in **S4** but whose reduction is not:

¹see some of the definitions of the terms used in this article in the appendix.

$$\begin{array}{c}
\frac{[\Box A]^1}{A} \quad [\neg A]^2_1 \\
\hline
\frac{\perp}{\neg \Box A} \\
\hline
\frac{\neg \Box A \rightarrow B}{B} \quad \neg B \\
\hline
\frac{\perp}{A}^2 \\
\hline
\frac{C \rightarrow A}{\Box(C \rightarrow A)}
\end{array}$$

Then, Medeiros [1] presented a new version for the **S4** system, the **NS4**. This new system is composed of the logical symbols $\wedge, \vee, \rightarrow, \perp$ and \Box and its rules are the usual, except for \Box -I, which is as follows:

$$\frac{[\Box \vec{B}]^k \quad \vdots \quad \frac{\Box \vec{B} \quad A}{\Box A}_k}{\Box A}_k$$

The restriction on \Box -I rule states that *all* the assumptions in $[\Box \vec{B}]^k$ must be discharged by the application of \Box -I and the premiss A must not depend on any assumption other than the ones in $\Box \vec{B}$.

With this new rule, we have the following reduction:

$$\frac{\frac{\vec{\Sigma} \quad \Box \vec{B}}{\Box A} \quad \frac{[\Box \vec{B}]^k \quad \Lambda_1 \quad A}{A}_k}{A} \triangleright \frac{\vec{\Sigma} \quad (\Box \vec{B}) \quad \Lambda_1}{A} \quad (3)$$

There is also the following permutative reduction.

$$\begin{array}{c}
 \begin{array}{c}
 \vec{\Sigma} \quad [\Box \vec{A}]^i \\
 \frac{\Box \vec{A} \quad B}{\Box B} \quad \Lambda_1 \quad \vec{\Psi} \quad [\Box B]^j [\Box \vec{D}]^k \\
 \frac{\Box B \quad \Box \vec{D}}{\Box C} \quad \Lambda_2 \quad C \quad \triangleright \\
 \hline
 \Box C \quad k, j
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{c}
 \vec{\Sigma} \quad \vec{\Psi} \quad [\Box \vec{A}]^k \\
 \frac{\Box \vec{A} \quad \Box \vec{D}}{\Box C} \quad \Lambda_1 \quad B \\
 \frac{\Box \vec{A} \quad B}{\Box B} \quad \Lambda_2 \quad C \quad [\Box \vec{D}]^i \\
 \hline
 \Box C \quad j, i
 \end{array}
 \end{array}
 \end{array} \quad (4)$$

4 A problem in the normalization proof of NS4

The proof of the normalization for NS4 by Medeiros [1] requires the use of the critical lemma. This lemma states that “if Π is a critical simplified derivation of C from Γ , then Π can be transformed into a simplified derivation Π' such that $I(\Pi') < I(\Pi)$.” The proof states that a derivation Π of C from Γ can be transformed in a simplified derivation Π_0 . And, by the critical lemma, a subderivation Σ of Π can be transformed in a subderivation Σ' such that $I(\Sigma') < I(\Sigma)$; then Π_1 is the derivation resulted from the substitution of Σ' for Σ in Π_0 . Medeiros aimed to show that $I(\Pi_1) < I(\Pi_0)$

Yuuki [3] pointed out two flaws on the proof of this lemma. The first one concerns critical derivations of the form

$$\Pi \equiv \frac{\frac{\frac{\Sigma_{0,1}}{F} \quad [\neg F]^i}{\perp} \quad \frac{\frac{\perp}{F} \quad i}{C} \quad \frac{\vec{\Sigma}}{\vec{H}} \quad r}{\Sigma_{0,2}} \quad (5)$$

where the major premiss F is the conclusion of \perp_C and r is an elimination

rule. According to Medeiros, this derivation can be transformed into

$$\Pi' \equiv \frac{\frac{\Sigma_{0,1} \vec{\Sigma}}{F \quad \vec{H}} \quad r}{\perp} \quad \text{or to} \quad \Pi'' \equiv \frac{\frac{\Sigma_{0,1} \vec{\Sigma}}{F \quad \vec{H}} \quad r}{C} \quad \frac{[\neg C]^i}{\perp} \quad \frac{\perp}{C} \quad i \quad (6)$$

depending on C being \perp or not.

Note that the assumptions of the form $\neg F$ discharged at the rule i may occur more than once in Π , multiplying the occurrences of the maximal premiss F in Π' and Π'' . In this case the index of either Π' or Π'' may be even greater than that of Π . Besides, one of the H_i 's in \vec{H} , say H_l , may be a maximal formula of degree $G(F)$ and, in this case, even if the F side connected with $\neg F$ is not a maximal formula in Π' , this H_l still is and the induction hypothesis cannot be used.

The second problem pointed out is when Π has degree $G(\Box A)$ and is a critical derivation of the form

$$\Pi \equiv \frac{\frac{\frac{\vec{\Sigma}}{\Box \vec{B}} \quad \Lambda_1}{\Box \vec{B} \quad A} \quad k}{\Box A} \quad \frac{[\neg \Box A]^i}{\perp} \quad \frac{\perp}{\Sigma_0} \quad \frac{\perp}{\Box A} \quad i \quad \frac{\frac{\vec{\Psi}}{\Box \vec{C}} \quad \Lambda_2}{B} \quad \frac{[\Box A]^j [\Box C]^\ell}{\ell, j} \quad \Box B \quad (7)$$

If $\Box A$ occurs more than once as top-formula of Λ_2 , by reducing Π to

$$\Pi' \equiv \frac{\frac{\frac{\vec{\Sigma}}{\Box \vec{B}} \quad \frac{\vec{\Psi}}{\Box \vec{C}}}{\Box B} \quad \frac{\frac{\frac{\frac{[\Box \vec{B}]^j}{\Box A} \quad \frac{\Lambda_1}{A}}{\Box A}^k \quad [\Box C]^\ell}{\Lambda_2}{B}^{\ell,j}}{\Box B} \quad [\neg \Box B]^i}{\frac{\perp}{\Sigma_0}} \quad \frac{\perp}{\Box B}^i$$

the number of occurrences of $\Box A$ as maximal formula in Π' will be greater than in Π .

Thus, it is possible that the reduction process generates copies of maximal formulas, so the index of Π' may be greater than that of Π .

5 Yet another proof of the critical lemma

Lemma 1 *A critical derivation of the form $\Pi \equiv \frac{\frac{\frac{[\neg F]^1}{\Sigma_1} \quad \vec{\Sigma}}{\frac{\perp}{F}^1} \quad \vec{B}}{C}^r$ where F*

is the conclusion of \perp_c , can be transformed in a derivation

$$\Pi_1 \equiv \frac{\frac{\frac{\frac{F}{\Sigma'_{1,1}} \quad [\neg F]^1}{\perp}}{\Sigma'_{1,2}} \quad \frac{\frac{\frac{\perp}{F}^1 \quad \vec{B}}{C}^r}{\Sigma'_{1,1}} \quad \vec{\Sigma}}{\perp} \quad \text{where } \Pi_1 \text{ is a simplified derivation without}$$

trivial formulas. Thus, the end-formula of $\Sigma'_{1,1}$ is not conclusion of \perp_c . Note that Π_1 has no more maximal formulas of degree equal to or higher than $G(F)$ than Π . We will use the symbol \propto to indicate the transformation of a derivation into a simplified derivation without trivial formulas.

Proof 1 *See the work of Medeiros [1].*

Theorem 5.1 *If Π is a critical simplified derivation of C from Γ , then Π can be transformed into a simplified derivation Π' such that $G(\Pi') < G(\Pi)$.*

Proof 2 *Suppose Π is a critical simplified derivation with maximal premisses of degree $G(\Pi)$ which are premisses of the last inference of Π , $\#G(\Pi)$ is the number of maximal formulas of Π with degree $G(\Pi)$ and $\ell(\Pi)$ is the length of Π . The proof is by induction on the pair $\langle \#G(\Pi), \ell(\Pi) \rangle$.*

$$1. \Pi \equiv \frac{\frac{\Sigma_1 \quad \Sigma_2}{A \quad B} \quad \frac{A \wedge B}{A}}{A} \triangleright \frac{\Sigma_1}{A} \equiv \Pi'$$

It is easy to see that $G(\Pi') < G(\Pi)$.

$$2. \Pi \equiv \frac{\frac{\frac{[A] \quad \Sigma_1}{\Sigma_2} \quad \frac{B}{A \rightarrow B}}{A} \quad \frac{A \rightarrow B}{B}}{B} \triangleright \frac{\frac{(A) \quad \Sigma_1}{\Sigma_2} \quad B}{B} \equiv \Pi'$$

It is easy to see that $G(\Pi') < G(\Pi)$.

$$3. \Pi \equiv \frac{\frac{\vec{\Sigma}}{\square \vec{B}} \quad \frac{\Lambda_1}{C}}{\frac{\square C}{C}} \triangleright \frac{\frac{\vec{\Sigma}}{(\square \vec{B})} \quad \Lambda_1}{C} \equiv \Pi'$$

If there existed a $\square B_l$ which is a maximal premiss at Π' , then it would be a maximal formula at Π and, as Π is a critical derivation, $G(\square B_l) < G(\square C)$. Thus, $G(\Pi') < G(\Pi)$.

$$4. \Pi \equiv \frac{\frac{\frac{\vec{\Sigma}}{\square \vec{B}} \quad \frac{\Lambda_1}{A}}{\square A} \quad \frac{\frac{[\square \vec{B}]^k \quad \vec{\Psi}}{\vec{H}} \quad \frac{[\square A]^j, [\vec{H}]^l}{\Lambda_2}}{\square C} \quad \frac{C}{C}}{\square C} \triangleright \frac{\frac{\frac{\vec{\Sigma}}{\square \vec{B}} \quad \vec{\Psi}}{\square A} \quad \frac{\frac{[\square \vec{B}]^l \quad \Lambda_1}{A}}{(\square A)} \quad \frac{[\vec{H}]^j}{C}}{\square C} \equiv \Pi_1$$

We have two cases to consider:

- (a) there is an occurrence of $\Box A$ which is top-formula of Λ_2 and major premiss of an application of \Box -E: *in this case, the number of maximal formulas of degree $G(\Box A)$ in Π_1 may be even greater than that of Π , as there may be more than one occurrence of $\Box A$ as top-formula of Λ_2 .*

There is a critical subderivation Ξ_1 of Π_1 of the form

$$\frac{\frac{\frac{[\Box \vec{B}]^k}{\Lambda_3} \quad A}{\Box A} \quad k}{A}$$

which can be reduced to $\Xi'_1 \equiv \frac{\Box \vec{B}}{\Lambda_3} \quad A$ (case 3).

- (b) there is an occurrence of $\Box A$ which is top-formula of Λ_2 and major premiss of \Box -I : *then, there is a critical subderivation Ξ_2 of the form*

$$\frac{\frac{\frac{[\Box \vec{B}]^k}{\Lambda_3} \quad A}{\Box A} \quad k \quad \frac{\frac{\vec{\Psi}}{\Box \vec{C}} \quad \frac{[\Box A]^l, [\Box \vec{C}]^j}{\Lambda_4} \quad D}{\Box D} \quad j, l}{\Box D}$$

The lenght of Ξ_2 is smaller than the lenght of Π . Hence, by induction hypothesis, we can reduce Ξ_2 to a Ξ'_2 such that $G(\Xi'_2) < G(\Pi)$.

Note that we cannot guarantee that the lenght of Ξ_2 is smaller than the lenght of Π if there were more than one occurrence of $\Box A$ as top-formula of Λ_2 in Π_1 , and if there were many occurrences of $\Box A$ as major premiss in Ξ_2 . That is the reason of the restriction on the beginning of the section.

Let Π_2 be the result of replacing each occurrence of critical subderivations of the form of Ξ_1 and the form of Ξ_2 in Π_1 by Ξ'_1 and Ξ'_2 respectively.

If $\Box A$ is the only major premiss that is maximal formula in Π , i.e., there is no member of \vec{H} which is a maximal premiss of the same degree of Π , then $G(\Pi_2) < G(\Pi)$ and $\Pi_2 = \Pi$. Otherwise, i.e., if there exists a k such that H_k is a maximal formula in Π , then $\#G(\Pi_2) < \#G(\Pi)$ and, as Π_2 is a critical derivation, by induction hypothesis Π_2 can be transformed into a derivation Π' such that $G(\Pi') < G(\Pi_2)$. Hence, as Π was transformed into Π_2 , $G(\Pi') < G(\Pi)$.

$$\begin{array}{c}
5. \Pi \equiv \frac{\frac{\frac{[\neg(A \wedge B)]}{\Sigma_1} \quad \perp}{A \wedge B} \quad A}{\perp} \quad \propto \quad \frac{\frac{\frac{\Sigma'_{1,1}}{A \wedge B} \quad [\neg(A \wedge B)]^1}{\perp} \quad \Sigma'_{1,2}}{\frac{\perp}{A \wedge B} \quad A} \quad \triangleright \quad \frac{\frac{\frac{\Sigma'_{1,1}}{A \wedge B} \quad [\neg A]^1}{\perp} \quad \Sigma'_{1,2}}{\frac{\perp}{A} \quad A} \\
\equiv \Pi_1
\end{array}$$

The \perp is major premiss in Π , so if the end formula of $\Sigma'_{1,1}$ is not the conclusion of an introduction rule, then the end-formula of $\Sigma'_{1,1}$ is not a maximal formula and $G(\Pi_1) < G(\Pi)$ and $\Pi_1 \equiv \Pi'$. If the end formula of $\Sigma'_{1,1}$ is the conclusion of an introduction rule, then Π_1 is of the

$$\begin{array}{c}
\frac{\frac{\frac{\Sigma_3}{A} \quad \Sigma_4}{A \wedge B} \quad A}{\perp} \quad [\neg A]^1 \quad \text{form} \quad \frac{\frac{\Sigma_3}{A} \quad [\neg A]^1}{\perp} \quad \Sigma'_{1,2} \\
\frac{\perp}{\Sigma'_{1,2}} \quad \frac{\perp}{A} \quad \text{which can be reduced to} \quad \frac{\perp}{A} \quad \frac{\perp}{A} \quad \text{I} \\
\equiv \Pi' \text{ and } G(\Pi') < G(\Pi).
\end{array}$$

$$\begin{array}{c}
6. \Pi \equiv \frac{\frac{\frac{[\neg(A \rightarrow B)]}{\Sigma_1} \quad \perp}{A \rightarrow B} \quad \Sigma_2}{B} \quad \propto \quad \frac{\frac{\frac{\Sigma'_{1,1}}{A \rightarrow B} \quad [\neg(A \rightarrow B)]^1}{\perp} \quad \Sigma'_{1,2}}{\frac{\perp}{A \rightarrow B} \quad A} \quad \triangleright \quad \frac{\frac{\perp}{A \rightarrow B} \quad \Sigma_2}{B} \quad A
\end{array}$$

$$\frac{\frac{\Sigma'_{1,1} \quad \Sigma_2}{A \rightarrow B} \quad A}{B} \quad \frac{[\neg B]^1}{\perp} \equiv \Pi_1$$

$$\frac{\perp}{\Sigma'_{1,2}}$$

$$\frac{\perp}{B} \text{ } ^1$$

If the end formula of $\Sigma'_{1,1}$ is not the conclusion of an introduction rule, then $G(\Pi_1) < G(\Pi)$ and $\Pi_1 \equiv \Pi'$.

If the end formula of $\Sigma'_{1,1}$ is the conclusion of an introduction rule,

$$\frac{\frac{\frac{[A]}{\Sigma_3} \quad \Sigma_4}{A \rightarrow B} \quad A}{B} \quad \frac{[\neg B]^1}{\perp} \text{ which can be re-}$$

$$\frac{\perp}{\Sigma'_{1,2}}$$

$$\frac{\perp}{B} \text{ } ^1$$

$$\frac{\Sigma_4 \quad (A) \quad \Sigma_3 \quad B \quad [\neg B]^1}{\perp} \equiv \Pi' \text{ and } G(\Pi') < G(\Pi).$$

$$\frac{\perp}{\Sigma'_{1,2}}$$

$$\frac{\perp}{B} \text{ } ^1$$

$$7. \Pi \equiv \frac{\frac{[\neg \Box A]}{\Sigma_1} \quad \frac{\frac{\perp}{\Box A}}{A}}{\perp} \quad \propto \quad \frac{\frac{\Sigma'_{1,1} \quad \Box A \quad [\neg \Box A]^1}{\perp}}{\Sigma'_{1,2}} \quad \triangleright \quad \frac{\frac{\Sigma'_{1,1} \quad \Box A}{A} \quad [\neg A]^1}{\perp} \equiv \Pi_1$$

$$\frac{\perp}{\Box A} \text{ } ^1$$

$$\frac{\perp}{A} \text{ } ^1$$

If the end formula of $\Sigma'_{1,1}$ is not the conclusion of an introduction rule, then $G(\Pi_1) < G(\Pi)$ and $\Pi_1 \equiv \Pi'$. If the end formula of $\Sigma'_{1,1}$ is the

conclusion of an introduction rule, then Π_1 is of the form

$$\begin{array}{c}
 \frac{\frac{\vec{\Psi}}{\Box \vec{B}} \quad \frac{\Lambda_1}{A}}{A}^k \quad \frac{\Box A}{A} \quad \frac{\perp}{\Sigma'_{1,2}} \quad [\neg A]^l \quad \text{which can be reduced to} \quad \frac{\frac{\vec{\Psi}}{(\Box \vec{B})} \quad \Lambda_1}{A} \quad \frac{\perp}{\Sigma'_{1,2}} \quad [\neg A]^l \\
 \frac{\perp}{A}^{k,l} \quad \frac{\perp}{A}^l \\
 \equiv \Pi_2
 \end{array}$$

If one of the $\Box B_i$ were a maximal formula in Π_2 , it would be a maximal formula in Π and, as Π is a critical derivation, $G(\Box B_i) < G(\Box A)$. Thus, $G(\Pi_2) < G(\Pi)$ and $\Pi_2 \equiv \Pi'$.

$$8. \Pi \equiv \frac{\frac{\frac{[\neg \Box A]^k}{\Sigma_1} \quad \frac{\perp}{\Box A}^k}{\Box A} \quad \frac{\vec{\Psi}}{\vec{H}} \quad \frac{[\Box A]^l, [\vec{H}]^j}{\Sigma_2} \quad C}{\Box C}^{l,j} \quad \propto$$

$$\frac{\frac{\Sigma'_{1,1}}{\Box A} \quad [\neg \Box A]^k}{(\perp)} \quad \frac{\frac{\perp}{\Box A}^k \quad \frac{\vec{\Psi}}{\vec{H}} \quad \frac{[\Box A]^l, [\vec{H}]^j}{\Sigma_2} \quad C}{\Box C}^{l,j} \quad \triangleright$$

$$\frac{\frac{\Sigma'_{1,1}}{\Box A} \quad \frac{\vec{\Psi}}{\vec{H}} \quad \frac{[\Box A]^l, [\vec{H}]^j}{\Sigma_2} \quad C}{\Box C}^{j,l} \quad \frac{[\neg \Box C]^k}{(\perp)} \quad \frac{\Sigma'_{1,2}}{\frac{\perp}{\Box C}^k} \equiv \Pi_1$$

Note that $\Sigma'_{1,1}$ is a subderivation of Σ_1 . Hence, if the subderivation $\Lambda \equiv$

$$\frac{\frac{\Sigma'_{1,1} \quad \vec{\Psi}}{\Box A} \quad \vec{H} \quad \Sigma_2}{\Box C} \quad \text{of } \Pi_1 \text{ is a critical derivation, its length is } l, j$$

smaller than the length of Π . Thus, by the induction hypothesis, Λ can be reduced to a derivation Λ' such that $G(\Lambda') < G(\Pi)$. The result of replacing each occurrence of Λ in Π_1 by Λ' is a derivation Π' such that $G(\Pi') < G(\Pi)$.

$$9. \Pi \equiv \frac{\frac{\frac{[\neg F]^1}{\Sigma_1} \quad \vec{\Psi}}{\frac{\perp}{F} \text{ }^1} \quad \vec{H}}{\perp} \text{ }_r \quad \propto \quad \frac{\frac{\Sigma'_{1,1} \quad \vec{\Psi}}{F} \quad [\neg F]^1}{\perp} \quad \Sigma'_{1,2} \quad \vec{\Psi} \quad \triangleright$$

$$\frac{\frac{\perp}{F} \text{ }^1}{\perp} \text{ }_r$$

$$\Pi_1 \equiv \frac{\frac{\Sigma'_{1,1} \quad \vec{\Psi}}{F} \quad \vec{H}}{\perp} \quad \Sigma'_{1,2} \quad \perp$$

The critical subderivation $\Lambda \equiv \frac{\Sigma'_{1,1} \quad \vec{\Psi}}{F} \quad \vec{H}$ of Π_1 is smaller than Π .

Thus, by the induction hypothesis, Λ can be reduced to a derivation Λ' such that $G(\Lambda') < G(\Pi)$. By replacing each occurrence of Λ in Π_1 by Λ' we achieve the desired derivation.

$$10. \Pi \equiv \frac{\frac{[\neg F]^1 \quad \Sigma'}{C} \quad F \rightarrow C}{\Sigma} \quad \triangleright \Pi_1 \equiv \frac{\frac{[F]^1}{B} \quad [\neg B]^2}{\frac{\perp}{\neg F} \text{ }^1} \quad \Sigma' \quad F \rightarrow C$$

$$\frac{\frac{\perp}{F} \text{ }^1}{\frac{B}{\text{ }^{\text{el}}}} \quad \frac{C}{\Sigma} \quad \frac{\perp}{B} \text{ }^2$$

If the \perp is a minor premiss of an application of E_{\rightarrow} , then $G(\Pi) < G(\Pi_1)$

A Definitions

Some definitions found here are based on the work of Medeiros [1].

Definition 1 *The premisses $(A \rightarrow B)$ of the rule (E_{\rightarrow}) , $(A \wedge B)$ of (E_{\wedge}) , $(A \vee B)$ of (E_{\vee}) , $(\Box A)$ of (E_{\Box}) , and the premisses $\Box B_1, \dots, \Box B_n$ of (I_{\Box}) are called major premisses and the others minor premisses.*

Definition 2 *A segment in a derivation is a sequence A_1, \dots, A_n of occurrences of the same formula in a branch such that A_1 is not the conclusion of an application of (E_{\vee}) nor a discharged assumption through an application of (I_{\Box}) , and A_n is not a minor premiss of (E_{\vee}) nor a major premiss of (I_{\Box}) .*

Definition 3 *The length of a segment is the number of formula occurrences in this segment.*

Definition 4 *A maximal segment in a derivation is a segment A_1, \dots, A_n such that A_1 is the conclusion of an application of an introduction rule or (\perp_c) , and A_n is a major premiss of an application of an elimination rule.*

Definition 5 *A maximal formula is a maximal segment whose length is 1(one). A premiss is called maximal premiss if it belongs to some maximal segment.*

Definition 6 *A formula A is a trivial formula if A is the conclusion of an application of (\perp_c) and the minor premiss of an application of (E_{\rightarrow}) whose major premiss is the assumption $\neg A$.*

Definition 7 *A derivation is a simplified derivation if the minor premisses of every application of (E_{\vee}) is the formula \perp .*

Definition 8 *The degree of a formula A , $G(A)$, is the number of occurrences in A of logical symbols different from \perp . The degree of a segment is the degree of the formula that belongs to this segment.*

Definition 9 *The degree of a derivation Π , $G(\Pi)$ is the highest degree of a maximal segment of Π . If Π does not have maximal segments, then $G(\Pi) = 0$.*

Definition 10 *A critical derivation is a derivation Π such that, if $G(\Pi) = d$, then the last inference of Π has a maximal premiss with degree d , and for every subderivation Σ of Π , $G(\Sigma) < G(\Pi)$.*

Definition 11 The index of a derivation Π is $I(\Pi) = \langle d, s \rangle$, where s is the sum of the lengths of the maximal segments of Π whose degree is d . If Π does not have maximal segments, then $I(\Pi) = \langle 0, 0 \rangle$.

Definition 12 A derivation Π is a normal derivation if Π does not have maximal segments.

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